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A BOUND ON THE DEGREE OF SCHEMES DEFINED BY QUADRATIC EQUATIONS

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ABSTRACT. We consider complex projective schemes $X \subset \mathbb{P}^r$ defined by quadratic equations and satisfying a technical hypothesis on the fibres of the rational map associated to the linear system of quadrics defining X. Our assumption is related to the syzygies of the defining equations and, in particular, it is weaker than properties N_2, $N_{2,2}$ and K_2. In this setting, we show that the degree, d, of $X \subset \mathbb{P}^r$ is bounded by a function of its codimension, c, whose asymptotic behaviour is given by $2^c/\sqrt[4]{\pi c}$, thus improving the obvious bound $d \leq 2^c$. More precisely, we get the bound $\binom{d}{2} \leq \binom{2c-1}{c-1}$. Furthermore, if X satisfies property N_p or $N_{2,p}$ we obtain the better bound $\binom{d+2-p}{2} \leq \binom{2c+3-2p}{c+1-p}$. Some classification results are also given when equality holds.

1. Introduction

The equations defining a projective variety (or scheme) and the syzygies among them play a central role in algebraic geometry. From this point of view, perhaps the most interesting case is that of quadratic equations.

In modern terms, the study of varieties defined by quadratic equations was initiated in Mumford's foundational paper [Mu], where it is proved that a multiple $|m\mathcal{L}|$ of any ample line bundle $\mathcal{L}$ over an algebraic variety X gives an embedding in a projective space which is defined by quadrics if m is big enough. Moreover, effective values of m were also obtained for curves and abelian varieties. Let us look more closely at the case of curves. A classical theorem of Castelnuovo [C] (also attributed to Mattuck [Ma] and Mumford [Mu]) states that a curve X of genus g embedded in projective space by a complete linear system $|\mathcal{L}|$ is projectively normal if $\deg(\mathcal{L}) \geq 2g+1$, and a theorem due to Fujita [F] and Saint-Donat [St.D], strengthening earlier work of Mumford, asserts that the homogeneous ideal of X is generated by quadrics if $\deg(\mathcal{L}) \geq 2g+2$. These results were generalized to syzygies by Green, proving that X satisfies property N_p if $\deg(\mathcal{L}) \geq 2g+1+p$ (see [G]). Since then, many efforts have been made to extend this type of result on property N_p to other varieties.

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Bearing in mind that *any* algebraic variety X admits an embedding into $\mathbb{P}^r$ such that its homogeneous ideal is generated by quadrics, a natural question arises:

What can be said about the degree of $X \subset \mathbb{P}^r$?

The aim of this paper is to obtain a bound on the degree, d, of a scheme $X \subset \mathbb{P}^r$ defined by quadrics in terms of its codimension c. It is obvious that $d \leq 2^c$, with equality if $X \subset \mathbb{P}^r$ is the complete intersection of c independent quadric hypersurfaces. In this case, the number of the defining equations is minimal with respect to c. On the other side, Zak showed in [Z] that if $X \subset \mathbb{P}^r$ is either a non-degenerate integral subvariety, or a finite set of points in general position, and the number of independent quadratic equations of X is almost maximal with respect to c, then $d \leq 2c$ (cf. Remark 2). But, to our best knowledge, very little is known about the degree of $X \subset \mathbb{P}^r$ if the number of the independent quadratic defining equations is neither minimal nor maximal with respect to c (cf. Remark 4).

However, our approach to this matter is a little bit different from that of [Z], even if we also use elementary techniques of projective geometry that allow us to work in a very general setting: let $\Lambda \subset H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(2))$ be a linear subspace of dimension $\alpha+1$ and let $X \subset \mathbb{P}^r$ denote the base scheme of Λ. Let $\Phi : \mathbb{P}^r \backslash X \to \mathbb{P}^\alpha$ be the morphism given by Λ. This map has been widely studied from many points of view (see, for instance, [V] and references therein). Our results are obtained under an assumption involving the map Φ. We impose an upper bound on the dimension of the set $W \subset G(1,r)$ consisting of lines $L \subset \mathbb{P}^r$ for which the restriction $\Phi_{|L} : L \to \Phi(L)$ is a double covering (see Lemma 2 and Definition 1). More precisely, we assume $\dim(W) \leq 2n+1$, where $n := \dim(X)$.

The main results of the paper are summarized in the following theorem. First, we introduce some terminology. We say that X is *reduced in codimension zero* if the scheme-theoretic intersection of X with a general linear subspace of $\mathbb{P}^r$ of codimension n is reduced, and we say that X is *smooth and integral in codimension one* if $n \geq 1$ and the scheme-theoretic intersection of X with a general linear subspace of $\mathbb{P}^r$ of codimension $n-1$ is a smooth integral curve.

Theorem 1. *Let $X \subset \mathbb{P}^r$ be a (possibly singular, non-reduced, reducible or non-equidimensional) complex projective scheme of degree d and dimension n defined by a linear system of quadrics $\Lambda \subset H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(2))$. Assume that X is reduced in codimension zero. Let $\alpha := \dim(\Lambda) - 1$ and let $c := r - n \geq 2$. Let $\Phi : \mathbb{P}^r \backslash X \to \mathbb{P}^\alpha$ be the morphism given by Λ and let $W \subset G(1,r)$ be the closure of the set of lines $L \subset \mathbb{P}^r$ for which the restriction $\Phi_{|L} : L \to \Phi(L)$ is a double covering. The following holds:*

(i) *if $\dim(W) \leq 2n+1$ then $\alpha \geq 2c-2$ and $\binom{d}{2} \leq \binom{2c-1}{c-1}$. Furthermore, if $\dim(W) \leq 2n$ then $\binom{d}{2} = \binom{2c-1}{c-1}$ if and only if $\alpha = 2c-2$.*

(ii) *if $\dim(W) \leq 2n$, $\binom{d}{2} = \binom{2c-1}{c-1}$ and, moreover, X is smooth and integral in codimension one then either $d = 3$, $c = 2$ and $g = 0$ or $d = 5$, $c = 3$ and $g = 1$, where g denotes the sectional genus of $X \subset \mathbb{P}^r$.*

We remark that our hypothesis is quite general. In fact, the assumption on W is closely related to the syzygies of the defining equations of X. For instance, if the trivial (or Koszul) relations among the elements of Λ are generated by linear syzygies, then the closure of any fibre of Φ is a linear space and $W = \emptyset$. This condition is called K_2 and it was introduced by Vermeire in [V]. Condition K_2 is weaker than the deeply studied property N_p defined in [G] (see [V]). In fact, for

any $p \geq 2$ we have

$$N_p \Rightarrow N_{2,p} \Rightarrow K_2 \Rightarrow W = \emptyset$$

(see [E-G-H-P] for the definition and results on property $N_{2,p}$). So, in particular, Theorem 1 yields the following:

Corollary 1. *Let $X \subset \mathbb{P}^r$ be a (possibly singular, non-reduced, reducible or non-equidimensional) complex projective scheme of degree d and dimension n defined by a linear system of quadrics $\Lambda \subset H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(2))$. Assume that X is reduced in codimension zero. Let $\alpha := \dim(\Lambda) - 1$ and let $c := r - n \geq 2$. If X satisfies condition K_2 then:*

(i) *$\alpha \geq 2c - 2$ and $\binom{d}{2} \leq \binom{2c-1}{c-1}$. Furthermore, $\binom{d}{2} = \binom{2c-1}{c-1}$ if and only if $\alpha = 2c - 2$.*

(ii) *if moreover X is smooth and integral in codimension one and $\binom{d}{2} = \binom{2c-1}{c-1}$, then either $d = 3$, $c = 2$ and $g = 0$ or $d = 5$, $c = 3$ and $g = 1$, where g denotes the sectional genus of $X \subset \mathbb{P}^r$.*

Thus, as an application of these results, we obtain some numerical conditions for a given variety (or scheme) to satisfy properties N_2, $N_{2,2}$ or K_2. For instance, as an immediate consequence of Corollary 1, we show in Example 2 that 8 general points in $\mathbb{P}^4$ cannot satisfy property N_2. This is an example where [G-L, Theorem 1] is sharp for $p = 2$. Furthermore, recent work of Han and Kwak shows that if $X \subset \mathbb{P}^r$ is a reduced scheme then property $N_{2,p}$ behaves well under inner projections (see [H-K]). Thanks to their result, we can improve Corollary 1 in the following way:

Corollary 2. *Let $X \subset \mathbb{P}^r$ be a (possibly singular, reducible or non-equidimensional) reduced complex projective scheme of degree d, dimension n and codimension c satisfying property N_p or $N_{2,p}$ for some $p \geq 2$. Then:*

(i) *$h^0(\mathbb{P}^r, \mathcal{I}_X(2)) \geq cp - \binom{p}{2}$ and $\binom{d+2-p}{2} \leq \binom{2c+3-2p}{c+1-p}$. Furthermore, $\binom{d+2-p}{2} = \binom{2c+3-2p}{c+1-p}$ if and only if $h^0(\mathbb{P}^r, \mathcal{I}_X(2)) = cp - \binom{p}{2}$.*

(ii) *if moreover X is smooth and integral in codimension one and $\binom{d+2-p}{2} = \binom{2c+3-2p}{c+1-p}$, then either $d = p + 1$, $c = p$ and $g = 0$ or $d = p + 3$, $c = p + 1$ and $g = 1$, where g denotes the sectional genus of $X \subset \mathbb{P}^r$.*

The main idea of the proof of Theorem 1 is the following. We choose a suitable smooth subvariety $Y \subset \mathbb{P}^r$ of dimension $c - 1$ and we estimate, *in two different ways*, the number of double points of the morphism $\Phi_{|Y} : Y \to \Phi(Y)$. This clearly explains why we need to assume $\dim(W) \leq 2n+1$. Otherwise, we cannot guarantee that the number of double points of the morphism $\Phi_{|Y} : Y \to \Phi(Y)$ is finite.

The paper is structured as follows. In Section 2 we fix notation. In Section 3 we obtain two results, on the number of common secant lines to a pair of smooth subvarieties $X, Y \subset \mathbb{P}^r$ and on the number of apparent double points of the 2-Veronese embedding of a smooth subvariety $Y \subset \mathbb{P}^r$, that we use in the sequel to compute the number of double points of $\Phi_{|Y}$. Finally, in Section 4 we include the proofs of Theorem 1 and Corollaries 1 and 2, as well as some related examples and remarks. Theorem 1 and Corollary 1 follow from Theorems 3 and 4, and Corollaries 5 and 6, respectively.

2. Notation

We work over the field of complex numbers. We will adopt the following notation:

$\mathbb{P}^r$: r-dimensional projective space
Λ: linear subspace of $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(2))$ generated by the elements $f_0, \ldots, f_\alpha$
α: $\dim(\Lambda) - 1$
X: subscheme of $\mathbb{P}^r$ defined by $f_0, \ldots, f_\alpha$, in brief: defined by Λ
d: degree of X in $\mathbb{P}^r$
n: dimension of X
c: codimension of X in $\mathbb{P}^r$
g: sectional genus of $X \subset \mathbb{P}^r$, if X is smooth and integral in codimension one
Φ: rational map from $\mathbb{P}^r$ to $\mathbb{P}^\alpha := \mathbb{P}(\Lambda^*)$, induced by the linear system Λ in $\mathbb{P}^r$
$N(r) := h^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(2)) - 1$
v_2: Veronese map from $\mathbb{P}^r$ to $\mathbb{P}^{N(r)}$ given by the complete linear system $|\mathcal{O}_{\mathbb{P}^r}(2)|$
$\langle Z \rangle$: linear span of a subvariety Z embedded in a projective space.

3. Preliminary results

Let us compute the number of common secant lines to smooth integral subvarieties $X, Y \subset \mathbb{P}^r$ in terms of the number of apparent double points of their respective linear sections. We prove a suitable generalization of the formula [G-H, p. 297] for two curves in $\mathbb{P}^3$.

We say that $X, Y \subset \mathbb{P}^r$ are *in general position* if the set of common secant lines to X and Y has the expected dimension.

Lemma 1. *Let X, Y be two smooth integral non-linear subvarieties in $\mathbb{P}^r$ of dimension h, k and degree d, δ, respectively, such that $h + k = r - 1$. Assume that X and Y are in general position in $\mathbb{P}^r$. For $i = 0, \ldots, h$, let a_i be the number of double points of a generic projection in $\mathbb{P}^{2i}$ of a generic linear section of X of dimension i (for $i = h$ we consider X, and for $i = 0$ we have $a_0 = \binom{d}{2}$). For $i = 0, \ldots, k$, let b_i be the number of double points of a generic projection in $\mathbb{P}^{2i}$ of a generic linear section of Y of dimension i (for $i = k$ we consider Y, and for $i = 0$ we have $b_0 = \binom{\delta}{2}$). Then the number of lines in $\mathbb{P}^r$ which are secant to both X and Y is $\sum_{i=0}^{\min\{h,k\}} a_i b_i$.*

Proof. Since $X, Y \subset \mathbb{P}^r$ are in general position and $h + k = r - 1$, we get a finite number $\varkappa$ of lines in $\mathbb{P}^r$ which are secant to both X and Y. We can assume that $h \leq k$. In this case we have to show that $\varkappa = \sum_{i=0}^{h} a_i b_i$.

Let G be the Grassmannian $G(1, r)$ of lines of $\mathbb{P}^r$, $\dim(G) = 2r - 2$. The secant lines of X give rise to a subvariety $S_X \subset G$ of dimension $2h$. The secant lines of Y give rise to a subvariety $S_Y \subset G$ of dimension $2k$. Obviously $\varkappa = S_X S_Y$ in the cohomology ring $H^*(G, \mathbb{Z})$ of G. It is well known that $H^*(G, \mathbb{Z})$ is generated by the cohomology classes of the so-called Schubert cycles $\Omega(p, q)$, where $\Omega(p, q)$ is the subvariety of G parametrizing the lines of $\mathbb{P}^r$ contained in a generic linear subspace B of dimension q and intersecting a generic linear subspace $A \subset B$ with $\dim(A) = p$. As $\dim[\Omega(p, q)] = p + q - 1$ we have

$$S_X = \sum_{i=0}^{h} \alpha_i \Omega(i, 2h + 1 - i)$$

$$S_Y = \sum_{j=0}^{h} \beta_j \Omega(2k + 1 - r + j, r - j)$$

for suitable $\alpha_i, \beta_j \in \mathbb{Z}$. By recalling that $h+k=r-1$ we can write

$$S_Y = \sum_{j=0}^{h} \beta_j \Omega(r-(2h+1)+j, r-j).$$

Now let us remark that $\Omega(i, 2h+1-i)\Omega(r-(2h+1)+j, r-j) = \delta_j^i$ (Kronecker symbols) so that $S_X S_Y = \sum_{i=0}^{h} \alpha_i \beta_i$. Moreover:

$\alpha_h = S_X \Omega(r-(h+1), r-h)$ is the number of secant lines to X contained in a generic linear subspace $B \subset \mathbb{P}^r$ of dimension $r-h$. As $\dim(X)=h$, this number is $\binom{d}{2} = a_0$.

$\alpha_{h-1} = S_X \Omega(r-(h+2), r-h+1)$ is the number of secant lines to X contained in a generic linear subspace $B \subset \mathbb{P}^r$ of dimension $r-h+1$ (cutting X along a generic section of dimension 1) and intersecting a generic linear subspace $A \subset B$ of dimension $r-h-2$, i.e. it is the number of double points of a generic projection in $\mathbb{P}^2$ of the curve $X \cap B$, hence $\alpha_{h-1} = a_1$. And so on until $\alpha_0 = a_h$.

For Y we can argue in the same way. $\square$

We now obtain the number of apparent double points of the 2-Veronese embedding of a smooth subvariety $Y \subset \mathbb{P}^r$ in terms of the number of apparent double points of the linear sections of Y.

Theorem 2. *Let Y be a smooth integral subvariety of $\mathbb{P}^r$ of dimension k and degree δ. Let $v_2(Y)$ be the 2-Veronese embedding of Y in $\mathbb{P}^{N(r)}$, where $N(r) := h^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(2)) - 1$. Let $\Delta[v_2(Y)]$ be the number of double points of a generic projection of $v_2(Y)$ in $\mathbb{P}^{2k}$. For $i=0,\dots,k$, let b_i be the number of double points of a generic projection in $\mathbb{P}^{2i}$ of a generic linear section of Y of dimension i (for $i=k$ we consider Y, and for $i=0$ we have $b_0 = \binom{\delta}{2}$). Then*

$$\Delta[v_2(Y)] = \sum_{i=0}^{k} \binom{2k+1}{k-i} b_i.$$

Proof. If $k=0$ we have nothing to prove. If $k \geq 1$, let $Y = Y_k \supset Y_{k-1} \supset \cdots \supset Y_1 \supset Y_0$ be a ladder of smooth hyperplane sections with $\dim(Y_{k-q}) = k-q$, $q = 0,1,\dots,k$. Let H be the class of the hyperplane divisor of Y. Let $\delta := H^k$ be the degree of Y. First of all we need a formula for the Segre classes of any Y_{k-q}. Let s_i be the i-th Segre class of Y, then we have

$$s_i(Y_{k-q}) = \sum_{j=0}^{q} \binom{q}{j} H^{q+j} s_{i-j} \text{ (with } s_{i-j}=0 \text{ if } i<j) \qquad (*).$$

By using [P-S, Theorem 3.4] we have

$$\begin{aligned}
b_k &= \tfrac{1}{2}\left(\delta^2 - \sum_{i=0}^{k} \binom{2k+1}{k-i} H^{k-i} s_i\right); \\
b_{k-1} &= \tfrac{1}{2}\left(\delta^2 - \sum_{i=0}^{k-1} \binom{2(k-1)+1}{k-1-i} H^{k-1-i} s_i(Y_{k-1})\right); \\
&\vdots \\
b_1 &= \tfrac{1}{2}\left(\delta^2 - \sum_{i=0}^{1} \binom{3}{1-i} H^{1-i} s_i(Y_1)\right); \\
b_0 &= \tfrac{1}{2}\left(\delta^2 - H^k s_0\right) = \binom{\delta}{2}.
\end{aligned}$$

On the other hand, the number of the double points of a generic projection of $v_2(Y)$ in $\mathbb{P}^{2k}$ is

$$\Delta[v_2(Y)] = \frac{1}{2}\left(2^{2k}\delta^2 - \sum_{i=0}^{k} \binom{2k+1}{k-i} 2^{k-i} H^{k-i} s_i\right).$$

Let us suppose that this number is equal to $a_k b_k + a_{k-1} b_{k-1} + \cdots + a_1 b_1 + a_0 b_0$ for a suitable choice of a_i and let us try to find a_i. For instance, by considering the coefficients of δ^2 we get

$$a_k + a_{k-1} + \cdots + a_1 + a_0 = 2^{2k} \qquad (**).$$

Moreover we must have

$$\sum_{i=0}^{k} \binom{2k+1}{k-i} 2^{k-i} H^{k-i} s_i = \sum_{i=0}^{k} a_{k-i} b'_{k-i} \qquad (***),$$

where $b'_{k-i} := \delta^2 - 2b_{k-i}$.

If we consider the coefficients of $H^{k-i}s_i$, $i = 0, 1, \ldots, k$, in $(***)$, taking care of the relations $(*)$, we get a system of $k+1$ linear equations in the $k+1$ unknowns a_i, $i = 0, 1, \ldots, k$, whose associated matrix is triangular, with a non-zero determinant.

We claim that the only solution of this system of linear equations is $a_i = \binom{2k+1}{k-i}$, $i = 0, 1, \ldots, k$. To see this fact we transform $(***)$ into

$$\sum_{i=0}^{k} \binom{2k+1}{k-i} (2^{k-i} - 1) H^{k-i} s_i = \sum_{i=1}^{k} a_{k-i} b'_{k-i}$$

and we write (by using the relations $(*)$)

$$\sum_{i=0}^{k} \binom{2k+1}{k-i} (2^{k-i} - 1) H^{k-i} s_i = \sum_{q=1}^{k} \sum_{i=0}^{k} \binom{2k+1}{k-i} \binom{k-i}{q} H^{k-i} s_i \qquad (i)$$

$$\sum_{i=1}^{k} \binom{2k+1}{i} b'_{k-i} = \sum_{q=1}^{k} \sum_{p=0}^{k-q} \sum_{j=0}^{q} \binom{2k+1}{q} \binom{2(k-q)+1}{k-q-p} \binom{q}{j} H^{k-(p-j)} s_{p-j} \qquad (ii)$$

where $s_{p-j} = 0$ when $p < j$. Now we fix any $q = 1, 2, \ldots, k$ and we look at the coefficients of $H^{k-i}s_i$, $i = 0, 1, \ldots, k$.

In (i) the coefficient of $H^{k-i}s_i$ is $\binom{2k+1}{k-i}\binom{k-i}{q}$. In (ii) it is

$$\binom{2k+1}{q} \sum_{j=0}^{\min(q,k-q-i)} \binom{2(k-q)+1}{k-q-i-j} \binom{q}{j} = \binom{2k+1}{q} \binom{2(k-q)+1+q}{k-q-i},$$

where the last equality is the well-known formula $\sum_{j=0}^{\min(a,b)} \binom{m}{b-j}\binom{a}{j} = \binom{m+a}{b}$.

Now it is easy to check that

$$\binom{2k+1}{k-i}\binom{k-i}{q} = \binom{2k+1}{q}\binom{2k-q+1}{k-q-i}.$$

Therefore $a_i = \binom{2k+1}{k-i}$, with $i = 0, 1, \ldots, k$, is the only solution of $(***)$ and obviously it is a solution for $(**)$ too. □

In particular, the following computation will be very useful for our purposes.

Corollary 3. *Let Y be a smooth k-dimensional quadric of $\mathbb{P}^r$. Let $v_2(Y)$ be the 2-Veronese embedding of Y in $\mathbb{P}^{N(r)}$. Then $\Delta[v_2(Y)] = \binom{2k+1}{k}$.*

Proof. Let us apply Theorem 2: in this case $b_0 = 1$ and $b_i = 0$ for $i = 1, \dots, k$ because the generic i-dimensional linear section of Y has no apparent double points. □

4. Main results

Let $\Phi : \mathbb{P}^r \dashrightarrow \mathbb{P}^\alpha$ be the rational map given by a linear system of quadrics Λ. Let us analyze the restriction of Φ to a line $L \subset \mathbb{P}^r$.

Lemma 2. *Let Λ be a linear system of quadrics in $\mathbb{P}^r$, let Φ be the associated rational map and let X be the scheme defined by Λ. Consider the restriction $\Phi_{|L}$ of the rational map Φ to a line L of $\mathbb{P}^r$. Then one of the following holds:*
(i) L is contained in X and $\Phi_{|L}$ is not defined;
(ii) L is a secant line for X and $\Phi_{|L}$ contracts L to a point;
(iii) L intersects X at one point and $\Phi_{|L}$ can be extended to an isomorphism among L and $\Phi(L)$;
(iv) $L \cap X = \emptyset$ and $\Phi_{|L}$ is a Veronese embedding of L;
(v) $L \cap X = \emptyset$ and $\Phi_{|L}$ is a double covering of $\Phi(L)$ by L.

Proof. Let us choose coordinates $(x : y)$ on L and let us consider that $\Phi_{|L}$ is given by a linear system of quadrics on $\mathbb{P}^1$ of the following type: $\lambda_i x^2 + \mu_i xy + \nu_i y^2$, $i = 0, \dots, \alpha$. In case (i) all quadrics vanish identically. In case (ii), let $(1 : 0)$ and $(0 : 1)$ be the coordinates of the two points $X \cap L$. All quadrics are of the following type: $\mu_i xy$, $i = 0, \dots, \alpha$, so that $\Phi(L) = (\mu_0 : \cdots : \mu_\alpha)$. In case ($iii$), let $(1 : 0)$ be the coordinates of $X \cap L$. All quadrics are of the following type: $\mu_i xy + \nu_i y^2 = (\mu_i x + \nu_i y)y$, $i = 0, \dots, \alpha$, so that $\Phi_{|L}$ can be extended to the isomorphism given by: $\mu_i x + \nu_i y$, $i = 0, \dots, \alpha$. In cases (iv) and (v), $\Phi_{|L}$ is a morphism given by a base point free linear system of degree two on L. If the linear system is complete, then $\Phi_{|L}$ is a Veronese embedding of $\mathbb{P}^1$. Otherwise, it is a double covering of $\mathbb{P}^1$. □

Lemma 3. *In the setting of Lemma 2, a line L such that $L \cap X = \emptyset$ yields case (v) if and only if it is a secant line to a non-linear fibre of Φ.*

Proof. Let $\Phi_P := \overline{\Phi^{-1}(\Phi(P))}$ be a non-linear fibre of Φ for some point $P \in \mathbb{P}^r \backslash X$. Let Q, Q' be any two points of Φ_P and let L be the line $\langle Q, Q' \rangle$. Let us assume that $L \cap X = \emptyset$ and let us consider $\Phi_{|L}$. As $\Phi(Q) = \Phi(Q')$, $\Phi_{|L}$ cannot be an embedding so that case (v) holds. Note that this is also true when $Q = Q'$, i.e. for tangent lines to Φ_P.

On the other hand, let L be a line in $\mathbb{P}^r$ such that $L \cap X = \emptyset$ and for which case (v) holds. Let $Q, Q' \in L$ be two points such that $\Phi(Q) = \Phi(Q')$. These points belong to some fibre Φ_P that cannot be a linear space, otherwise L would be contained in Φ_P and case (ii) would hold. Hence L is a secant line for Φ_P. □

Definition 1. In the setting of Lemma 2, let us consider the set $\mathcal{W} \subset G(1, r)$ consisting of lines L in $\mathbb{P}^r$ for which case (v) holds. Let W be the Zariski closure of $\mathcal{W}$.

Corollary 4. *$W = \emptyset$ if and only if all fibres of Φ are linear spaces.*

Proof. This is just a reformulation of Lemma 3. □

Now we can prove the main results of the paper:

Theorem 3. *Let $X \subset \mathbb{P}^r$ be a scheme of degree d and dimension n defined by a linear system of quadrics Λ. Assume that X is reduced in codimension zero. Let $\alpha := \dim(\Lambda) - 1$ and let $c := r - n \geq 2$. If $\dim(W) \leq 2n + 1$, then $\alpha \geq 2c - 2$ and $\binom{d}{2} \leq \binom{2c-1}{c-1}$. Furthermore, if $\dim(W) \leq 2n$ then $\binom{d}{2} = \binom{2c-1}{c-1}$ if and only if $\alpha = 2c - 2$.*

Proof. Let $\Phi : \mathbb{P}^r \dashrightarrow \mathbb{P}^\alpha$ be the rational map given by Λ, and let $v_2 : \mathbb{P}^r \to \mathbb{P}^{N(r)}$ be the 2-Veronese embedding of $\mathbb{P}^r$. Then $\Phi = \pi \circ v_2$, as rational maps, where $\pi : \mathbb{P}^{N(r)} \dashrightarrow \mathbb{P}^\alpha$ is the projection from the linear subspace $B := \langle v_2(X)\rangle$ of codimension $\alpha + 1$ in $\mathbb{P}^{N(r)}$ and $N(r) := \binom{r+2}{2} - 1$. Furthermore, we remark that $B \cap v_2(\mathbb{P}^r) = v_2(X)$ as X is defined by Λ.

Let us choose a generic c-dimensional linear space $A \subset \mathbb{P}^r$ such that $A \cap X$ is given by d distinct points $x_1, \dots, x_d$. This is possible since we assume X is reduced in codimension zero. Let $Y \subset A$ be a smooth quadric of dimension $c - 1$ such that $Y \cap X = \emptyset$. Then $\Phi_{|Y}$ is a regular map. We claim that $\Phi_{|Y} : Y \to \Phi(Y)$ is a finite morphism. If $\Phi_{|Y}$ has a positive dimensional fibre over a point $R \in \mathbb{P}^\alpha$, then we get also a positive dimensional fibre of $\pi_{|v_2(Y)}$ over the same point R, but this fibre is contained in $\langle B, R\rangle$, then it intersects B. Hence we would have $v_2(Y) \cap B \neq \emptyset$, which is a contradiction because $\emptyset = X \cap Y = v_2(X) \cap v_2(Y) = B \cap v_2(\mathbb{P}^r) \cap v_2(Y) = B \cap v_2(Y)$. This proves the claim.

We now claim that $\Phi_{|Y}$ cannot have infinitely many fibres containing two or more points. In fact, let Q, Q' be two points of Y such that $\Phi(Q) = \Phi(Q')$ and let $L \subset A$ be the line $\langle Q, Q'\rangle$. It follows from Lemma 2 that either case (ii) or case (v) holds for L. The common secant lines to X and Y coincide with the secant lines to $X \cap A$. Since X is defined by quadrics $X \cap A$ does not contain three points on a line, so the number of common secant lines to X and Y is equal to $\binom{d}{2}$. We now prove that for generic $Y \subset A \simeq \mathbb{P}^c$ there are at most a finite number of secant lines $\langle Q, Q'\rangle$, with $Q, Q' \in Y$ and $\Phi(Q) = \Phi(Q')$, for which Lemma 2 (v) holds. It is here where we strongly use the assumption $\dim(W) \leq 2n + 1$. Consider the rational map $\psi : \mathbb{P}^r \times \mathbb{P}^r \dashrightarrow G(1, r)$ given by $\psi(Q, Q') = \langle Q, Q'\rangle$. Let us define $V := \{(Q, Q') \in \mathbb{P}^r \times \mathbb{P}^r \mid \psi(Q, Q') \in W, \Phi(Q) = \Phi(Q')\}$. Then $\dim(V) \leq 2n+2$ as $\dim(W) \leq 2n+1$. Therefore, for generic Y, in $\mathbb{P}^r \times \mathbb{P}^r$ we have $\dim[(Y \times Y) \cap V] \leq 0$, since $\dim(V) + \dim(Y \times Y) \leq 2n + 2 + 2c - 2 = 2r = \dim(\mathbb{P}^r \times \mathbb{P}^r)$. This proves the claim.

It follows that $\Phi(Y) \subset \mathbb{P}^\alpha$ has only a finite number $\eta(Y)$ of singular points. In particular, B intersects the secant variety of $v_2(Y)$ in $\mathbb{P}^{N(r)}$ in a finite number of points. Since the dimension of the secant variety of $v_2(Y)$ in $\mathbb{P}^{N(r)}$ is the expected one, $2\dim[v_2(Y)] + 1 = 2c - 1$, we get

$$2c - 1 = \dim(\sec[v_2(Y)]) \leq \operatorname{codim}(B) = \alpha + 1$$

whence $\alpha \geq 2c - 2$, proving the first statement.

Moreover, the number of singular points of $\pi[v_2(Y)]$ is bounded by $\Delta[v_2(Y)]$ so we have

$$\binom{d}{2} \leq \eta(Y) \leq \Delta[v_2(Y)] = \binom{2c-1}{c-1},$$

where the equality follows from Corollary 3.

Furthermore, if $\dim(W) \leq 2n$ then $\dim(V) \leq 2n+1$ and hence no double point of $\Phi(Y)$ comes from a line L for which Lemma 2 (v) holds, arguing as before. Therefore

$\binom{d}{2} = \eta(Y)$. On the other hand, $\eta(Y) = \Delta[v_2(Y)]$ if and only if $\operatorname{codim}(B) = \dim(\sec[v_2(Y)])$, that is, if and only if $\alpha = 2c-2$. $\square$

Remark 1. The bound $\binom{d}{2} \le \binom{2c-1}{c-1}$ is better than the obvious bound $d \le 2^c$. In fact, a simple calculation shows that our bound is given asymptotically by $d \le \frac{2^c}{\sqrt[4]{\pi c}}$.

Remark 2. If $X \subset \mathbb{P}^r$ is a non-degenerate integral subvariety (resp. a finite set of points in general position) defined by quadrics then it follows from [Z, Corollary 5.4] that $c \le \alpha + 1 \le \binom{c+1}{2}$. On the one hand, if $c \le \alpha + 1 < 2c-1$ then $\dim(W) > 2n+1$ by Theorem 3, so our method say nothing about d. On the other hand, if $\binom{c}{2} < \alpha + 1 \le \binom{c+1}{2}$ then $d \le 2c$, and $\alpha + 1 = \binom{c+1}{2}$ if and only if $d = c+1$ and $X \subset \mathbb{P}^r$ is a variety of minimal degree (i.e. either a cone over the Veronese surface $v_2(\mathbb{P}^2) \subset \mathbb{P}^5$ or a rational normal scroll) (cf. [Z, Proposition 5.6, Corollary 5.8 and Remark 5.9]).

Remark 3. In view of Remark 2, our result is more relevant in the wide range $2c-2 \le \alpha < \binom{c}{2}$. Furthermore, looking at the proof of Theorem 3 one observes that the closer is α to $2c-2$, the better should be the bound $\binom{d}{2} \le \binom{2c-1}{c-1}$. It would be very interesting to find, under similar hypotheses, a bound on the degree of X involving not only c but also α. For instance, under the assumptions of Theorem 3 it can be shown that $\binom{d}{2} \le \binom{2c-1}{c-1} + 2c - 2 - \alpha$ if moreover $\Phi(Y)$ is non-degenerate in $\mathbb{P}^\alpha$. More generally, $\binom{d}{2} \le \binom{2c-1}{c-1} + 2c - 2 - \beta$, where β denotes the dimension of the linear span of $\Phi(Y)$ in $\mathbb{P}^\alpha$. However, these bounds are probably far from being optimal.

Remark 4. A bound on the degree of a zero dimensional scheme defined by quadrics was conjectured in [E-G-H, Conjecture $(II_{m,r})$]. In the particular case $2c-2 = \alpha$, where our bound turns out to be stronger, Conjecture $(II_{m,r})$ predicts $d \le 2^{c-1} + 1$. We would like to remark that we actually get the better bound $\binom{d}{2} \le \binom{2c-1}{c-1}$ under the extra assumption $\dim(W) \le 1$.

In Theorem 3, we assume X is reduced in codimension zero and $\dim(W) \le 2n+1$. This is crucial to prove that $\Phi_{|Y} : Y \to \Phi(Y)$ has only finitely many double points. Let us see that these hypotheses cannot be dropped in the following two remarks.

Remark 5. In the proof of Theorem 3, we assume X is reduced in codimension zero to ensure that $X \cap A$ consists of d different points. In this way, we have finitely many common secant lines to X and Y, whence $\Phi(Y)$ has finitely many double points coming from lines as in Lemma 2 (ii). This is no longer true if $X \cap A$ is non-reduced, as the following example shows. Consider $\Lambda \subset H^0(\mathbb{P}^c, \mathcal{O}_{\mathbb{P}^c}(2))$ generated by $X_i X_j$ for $1 \le i \le j \le c$. Then $X \subset \mathbb{P}^c$ is supported at the point $(1 : 0 : \cdots : 0)$. In this case, every line passing through $(1 : 0 : \cdots : 0)$ is contracted by $\Phi(Y)$. Hence, for every smooth quadric Y in $\mathbb{P}^c$ of dimension $c-1$ not passing through $(1 : 0 : \cdots : 0)$ the morphism $\Phi_{|Y} : Y \to \Phi(Y)$ is a double covering of $\Phi(Y) = v_2(\mathbb{P}^{c-1}) \subset \mathbb{P}^{N(c-1)}$ by Y.

Remark 6. On the other hand, the assumption on W is used to guarantee that $\Phi(Y)$ has finitely many double points coming from lines as in Lemma 2 (v). Unfortunately, this condition cannot be relaxed. Let us consider the following example. Choose coordinates $(x : y : z : u : w)$ in $\mathbb{P}^4$ and fix the hyperplane $w = 0$. Let F_0, F_1, F_2 be three generic degree two forms in $\mathbb{C}[x, y, z, u]$ such that the intersection

of the corresponding quadrics is given by 8 distinct points in the hyperplane $w = 0$. Let $\Phi : \mathbb{P}^4 \dashrightarrow \mathbb{P}^6$ be the rational map given by $(F_0 : F_1 : F_2 : wx : wy : wz : wu)$ and let X be the base locus of the linear systems Λ of quadrics giving Φ. X is the union of the 8 points and $(0 : 0 : 0 : 0 : 1)$, so that $r = 4$, $\alpha = 6$, $d = 9$, $n = 0$ and $c = 4$. It is easy to see that $\dim(Z) = 4$, where $Z := \Phi(\mathbb{P}^4)$, and the fibre over any point of Z is a point with the exception of points $(h : k : l : 0 : 0 : 0 : 0)$ whose fibres are positive dimensional, and the generic fibre is an elliptic smooth quartic of $\mathbb{P}^3$, intersection of two quadrics. Here $\dim(W) \geq 2$ by Lemma 3, and Theorem 3 does not hold since $36 = \binom{9}{2} > \binom{7}{3} = 35$.

Remark 7. In the proof of Theorem 3 we showed that $\Phi_{|Y} : Y \to \Phi(Y)$ is finite. Moreover, since $\dim(W) \leq 2n + 1$ and $A := \langle Y \rangle \subset \mathbb{P}^r$ is generic we deduce that $\dim(W \cap \mathbb{G}(1, A)) \leq 1$. So it easily follows from Lemma 3 that $\Phi_{|A} : A \dashrightarrow \Phi(A)$ is birational (in particular, if $n = 0$ the assumptions of Theorem 3 imply that $\Phi : \mathbb{P}^r \dashrightarrow \mathbb{P}^\alpha$ is birational onto its image). Hence $\dim(Z) \geq \dim(\Phi(A)) \geq c$, where $Z := \Phi(\mathbb{P}^r)$. As $\dim(Z) = \rho - 1$, where ρ is the generic rank of the Jacobian matrix of Λ, it follows that a necessary condition to get $\dim(W) \leq 2n+1$ is $\rho \geq c+1$. Note that, in concrete examples, to determine ρ is easier than to estimate the dimension of W.

In practice, it could be difficult to compute the dimension of W. However, as we pointed out in the introduction, we have $W = \emptyset$ as soon as condition K_2 holds.

Corollary 5. *Let $X \subset \mathbb{P}^r$ be a scheme of degree d and dimension n defined by a linear system of quadrics Λ. Assume that X is reduced in codimension zero. Let $\alpha := \dim(\Lambda) - 1$ and let $c := r - n \geq 2$. If X satisfies condition K_2 then $\alpha \geq 2c - 2$ and $\binom{d}{2} \leq \binom{2c-1}{c-1}$. Furthermore, $\binom{d}{2} = \binom{2c-1}{c-1}$ if and only if $\alpha = 2c - 2$.*

Proof. If X, or Λ, satisfies condition K_2 then the restriction $\Lambda_{|L}$ to a line $L \subset \mathbb{P}^r$ also satisfies K_2 (see [V, Lemma 4.2])). Therefore, if $L \cap X = \emptyset$ then $\Lambda_{|L}$ is given by the complete linear system of quadrics, so that $W = \emptyset$. Hence we can apply Theorem 3. □

Remark 8. According to Theorem 3, the inequality $\alpha + 1 \geq 2c - 1$ is a necessary condition to have $\dim(W) \leq 2n + 1$. As far as we know, this bound on the number of quadrics defining X was not known even for schemes satisfying property N_2 (cf. [H-K, Corollary 3.7 and Remark 3.9]).

Example 1. Generic linear sections of (a cone over) either $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ or $G(1, 4) \subset \mathbb{P}^9$ are examples of varieties satisfying N_2 and the equalities obtained in Theorem 3 and Corollary 5 with (d, c) equal to either $(3, 2)$ or $(5, 3)$, respectively (cf. Remark 9).

Example 2. Let X be the scheme given by 8 points in general position in $\mathbb{P}^4$. X is the base locus of a generic 6-dimensional linear system of quadrics Λ, and property N_1 holds (see [G-L, Theorem 1]). Here $c = 4$, $\alpha = 6$ and $d = 8$. Then $\alpha = 2c - 2$ but $\binom{8}{2} \neq \binom{7}{3}$, whence X does not satisfy property N_2 by Corollary 5. This shows an example where [G-L, Theorem 1] is sharp for $p = 2$.

The same computation shows that a curve of genus three embedded in $\mathbb{P}^5$ with degree 8 does not satisfy property N_2. In that case, it easily follows also from [G-L, Theorem 2].

Remark 9. The set of pairs of integers (d,c) satisfying the Diophantine equation $\binom{d}{2} = \binom{2c-1}{c-1}$ is not known (see for instance [dW]). The pairs $(3,2)$, $(5,3)$ and $(221,9)$ are solutions, but there could be other ones. However, if the generic curve section of X is smooth and integral, $(3,2)$ and $(5,3)$ are the only possibilities for equality in Theorem 3 and Corollary 5, thanks to the following:

Theorem 4. *In Theorem 3, if* $\dim(W) \leq 2n$, $\binom{d}{2} = \binom{2c-1}{c-1}$ *and, moreover, X is smooth and integral in codimension one then either $d=3$, $c=2$ and $g=0$ or $d=5$, $c=3$ and $g=1$.*

Proof. We consider a generic smooth integral subvariety $Y \subset \mathbb{P}^r$ of dimension $c-1$, disjoint from X, such that $\langle Y \rangle =: A \simeq \mathbb{P}^{c+1}$. Consider $\Phi(Y) \subset \mathbb{P}^\alpha$. The double points of $\Phi(Y)$ correspond to lines as in cases (ii) and (v) of Lemma 2. Since $\dim(W) \leq 2n$, one gets as in Theorem 3 that actually no double point of $\Phi(Y)$ comes from a line as in case (v). Hence $\Phi(Y)$ has only a finite number $\eta(Y)$ of double points, and $\eta(Y)$ is the number of common secant lines to X and Y, or equivalently, the number of common secant lines to $X \cap A$ and Y. As $Y \subset A$ is generic and $X \cap A$ is a smooth integral curve by hypothesis, we can compute the number of common secant lines to $X \cap A$ and Y by using Lemma 1. On the other hand, since $\binom{d}{2} = \binom{2c-1}{c-1}$ it follows from Theorem 3 that $\alpha = 2c-2$. Therefore $\eta(Y) = \Delta[v_2(Y)]$, and the second number can be computed by Theorem 2. As $a_0 = \binom{d}{2}$, $a_1 = \binom{d-1}{2} - g$ and $b_i = 0$ for $i \geq 2$ because the generic i-dimensional linear section of Y has no apparent double points, these two computations yield

$$\tbinom{d}{2}b_0 + [\tbinom{d-1}{2} - g]b_1 = \eta(Y) = \tbinom{2c-1}{c-1}b_0 + \tbinom{2c-1}{c-2}b_1.$$

Since $\binom{d}{2} = \binom{2c-1}{c-1}$ we deduce $\binom{d-1}{2} - g = \binom{2c-1}{c-2}$. By using Castelnuovo's inequality in $\mathbb{P}^{c+1}$ we have

$$\tfrac{1}{2}(d^2-3d+2) - \tbinom{2c-1}{c-2} = g \leq \tfrac{d^2-2d+1}{2c} + \tfrac{d-1}{2}$$

(see [A-C-G-H] p. 116). As

$$\tbinom{2c-1}{c-2} = \tfrac{c-1}{c+1}\tbinom{2c-1}{c-1} = \tfrac{c-1}{c+1}\tbinom{d}{2},$$

we get

$$\tfrac{1}{2}(d^2-3d+2) - \tfrac{c-1}{c+1}\tbinom{d}{2} \leq \tfrac{d^2-2d+1}{2c} + \tfrac{d-1}{2},$$

i.e. $d - 2 - \frac{c-1}{c+1}d \leq \frac{d-1}{c} + 1$, and, equivalently, $\frac{2d-2c-2}{c+1} \leq \frac{d+c-1}{c}$. So we get $d \leq \frac{3c^2+2c-1}{c-1}$. From the relation $\binom{2c-1}{c-1} = \binom{d}{2}$ we have

$$2\tbinom{2c-1}{c-1} \leq \tfrac{3c^2+2c-1}{c-1}\left(\tfrac{3c^2+2c-1}{c-1} - 1\right),$$

and it is easy to see that this last inequality cannot be satisfied for $c \geq 6$. If $c \leq 5$, taking care of the relation $\binom{d}{2} = \binom{2c-1}{c-1}$, then the only possibilities are $d=3$, $c=2$, $g=0$ and $d=5$, $c=3$, $g=1$. □

Remark 10. If X is smooth and integral in codimension one and $\dim(W) \leq 2n+1$, then Lemma 1 and Theorem 2 yield

$$a_0b_0 + a_1b_1 \leq \eta(Y) \leq \tbinom{2c-1}{c-1}b_0 + \tbinom{2c-1}{c-2}b_1.$$

Consider $Y := Y_e \subset \mathbb{P}^{c+1}$ the complete intersection of two hypersurfaces of degree e. Then $b_0(Y_e) = \binom{e^2}{2}$, and $b_1(Y_e) = \binom{e^2-1}{2} - (e^2(e-2)+1)$ since the sectional genus of Y_e is $e^2(e-2)+1$. Therefore, for every $e \geq 2$ we have the relation $a_0\binom{e^2}{2} + a_1[\binom{e^2-1}{2} - (e^2(e-2)+1)] \leq \binom{2c-1}{c-1}\binom{e^2}{2} + \binom{2c-1}{c-2}[\binom{e^2-1}{2} - (e^2(e-2)+1)]$. Dividing by $\binom{e^2}{2}$ we get $a_0 + a_1 f(e) \leq \binom{2c-1}{c-1} + \binom{2c-1}{c-2} f(e)$, where $f(e) = \frac{\binom{e^2-1}{2} - (e^2(e-2)+1)}{\binom{e^2}{2}}$. Since $\lim_{e\to\infty} f(e) = 1$ we get $a_0 + a_1 \leq \binom{2c-1}{c-1} + \binom{2c-1}{c-2}$. So we obtain the bound

$$\binom{d}{2} + \binom{d-1}{2} - g \leq \binom{2c-1}{c-1} + \binom{2c-1}{c-2},$$

where g denotes the sectional genus of $X \subset \mathbb{P}^r$.

Remark 11. This bound is slightly better than the bound obtained in Theorem 3. Furthermore, if X is smooth and integral in higher codimensions the same argument produces (a little bit) stronger and stronger bounds involving numerical characters of the corresponding linear section.

Remark 12. A priori, the choice of Y in Theorem 3 and Remark 10 might seem rather arbitrary. However, in Theorem 3 the same bound is obtained by taking a hypersurface $Y \subset \mathbb{P}^c$ of any degree. On the other hand, in Remark 10 it is natural to consider a complete intersection in view of Hartshorne's Conjecture, and the bound does not change if we choose a complete intersection of hypersurfaces of different degrees. Furthermore, some computations where $Y \subset \mathbb{P}^{c+1}$ is a non complete intersection codimension two subvariety, for $c+1 \leq 5$, suggest that it is not possible to improve the bound of Remark 10 with our method.

Corollary 6. *In Corollary 5, if moreover X is smooth and integral in codimension one and $\binom{d}{2} = \binom{2c-1}{c-1}$, then either $d = 3$, $c = 2$ and $g = 0$ or $d = 5$, $c = 3$ and $g = 1$.*

Proof. This is an immediate consequence of Theorem 4. □

Furthermore, in view of [H-K], Corollaries 5 and 6 yield a stronger result when $X \subset \mathbb{P}^r$ satisfies property N_p or $N_{2,p}$ for $p \geq 3$:

Proof of Corollary 2. If $p \geq 3$, let $r' := r + 2 - p$ and let $X' \subset \mathbb{P}^{r'}$ denote the inner projection from $p-2$ general smooth points of $X \subset \mathbb{P}^r$. Let $d' := d + 2 - p$ and $c' := c+2-p$ denote, respectively, the degree and codimension of $X' \subset \mathbb{P}^{r'}$. If $X \subset \mathbb{P}^r$ satisfies property N_p or $N_{2,p}$ then $h^0(\mathbb{P}^r, \mathcal{I}_X(2)) \geq cp - \binom{p}{2}$ by [H-K, Corollary 3.7], and $X' \subset \mathbb{P}^{r'}$ satisfies property $N_{2,2}$ by [H-K, Corollary 3.3]. Therefore $\binom{d'}{2} \leq \binom{2c'-1}{c'-1}$ by Corollary 5, and hence $\binom{d+2-p}{2} \leq \binom{2c+3-2p}{c+1-p}$. Furthermore, $\binom{d+2-p}{2} = \binom{2c+3-2p}{c+1-p}$ if and only if $h^0(\mathbb{P}^{r'}, \mathcal{I}_{X'}(2)) = 2c' - 1$ by Corollary 5. Note that this happens if and only if $h^0(\mathbb{P}^r, \mathcal{I}_X(2)) = cp - \binom{p}{2}$ (cf. [H-K, Proposition 3.5]). This proves (i). Furthermore, if $X \subset \mathbb{P}^r$ is smooth and integral in codimension one we claim that also $X' \subset \mathbb{P}^{r'}$ is smooth and integral in codimension one. By induction, it is enough to prove the claim for the inner projection of $X \subset \mathbb{P}^r$ from a single point. Let $x \in X$ be a smooth general point and let $C \subset \mathbb{P}^{c+1}$ denote a smooth integral curve section of $X \subset \mathbb{P}^r$ passing through x. Let $C' \subset \mathbb{P}^{c'+1}$ denote the inner projection of $C \subset \mathbb{P}^{c+1}$ from x. Then $C' \subset \mathbb{P}^{c'+1}$ is a smooth integral curve section of $X' \subset \mathbb{P}^{r'}$ isomorphic to $C \subset \mathbb{P}^{c+1}$ since there are no trisecant lines to $C \subset \mathbb{P}^{c+1}$ passing through x, as $X \subset \mathbb{P}^r$ is defined by quadrics. Therefore if

$\binom{d+2-p}{2} = \binom{2c+3-2p}{c+1-p}$, i.e. if $\binom{d'}{2} = \binom{2c'-1}{c'-1}$, then either $d' = 3$, $c' = 2$ and $g' = 0$ or $d' = 5$, $c' = 3$ and $g' = 1$ by Corollary 6, that is, either $d = p+1$, $c = p$ and $g = 0$ or $d = p+3$, $c = p+1$ and $g = 1$. This proves (ii). $\square$

Remark 13. On the other hand, as it is well known, both rational normal curves of degree $p+1$ and elliptic normal curves of degree $p+3$ satisfy property N_p (see, for instance, [G] or [G-L]).

Remark 14. The bounds obtained in Corollary 2 improve those of Theorem 1 and Corollary 1. In fact, d is bounded asymptotically by $\frac{2^{c+2-p}}{\sqrt[4]{\pi(c+2-p)}}$.

According to Remark 9, it would be interesting to answer the following:

Question 1. Is it possible to obtain 221 points in $\mathbb{P}^9$ as the base locus of a linear system of quadrics satisfying property N_2, $N_{2,2}$, K_2 or $\dim(W) \leq 0$?

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